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CHARACTERIZATION AND EXISTENCE RESULTS OF SOLUTIONS IN NON-CONVEX CONTROLLED MINIMIZATION MODELS*

Savin Treanţă† Marilena Ciontescu‡ Vivek Laha§
Fangfang Shi¶

Dedicated to the memory of Professor Haim Brezis

Abstract

This paper investigates the efficiency conditions and a dual model for a non-convex multi-cost minimization problem. Taking into account that the concept of (χ, ϱ)-invexity extends many types of convexity, we study and describe the solution set for this non-convex multi-cost minimization problem under assumptions of (χ, ϱ)-invexity, (χ, ϱ)-pseudoinvexity, or (χ, ϱ)-quasiinvexity for the involved multiple integral functionals.

Keywords: $(\chi,\ \varrho)$-invexity, non-convex multi-cost extremization problem, efficiency criteria, dual model.
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†savin.treanta@upb.ro, (1) Department of Applied Mathematics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania; (2) Academy of Romanian Scientists, 54 Splaiul Independentei, 050094 Bucharest, Romania; (3) Fundamental Sciences Applied in Engineering - Research Center (SFAI), National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania

‡marilena.ciontescu@upb.ro, Department of Applied Mathematics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania

§laha.vivek333@gmail.com, Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, 221005, U.P, India

¶15298387799@163.com, School of Mathematics, Hohai University, Nanjing 210098, China

1 Introduction

The subject addressed in this article is a developing area of operations research that plays an important role in the study of multi-dimensional optimization problems. The aim of this paper is to provide some characterization and existence results of solutions in non-convex controlled multi-objective minimization models. Over time, many scientists have contributed to this field. In 1995, Bhatia and Kumar [3] investigated a multiobjective variational control problem. The definition of invexity for continuous functions has been extended to ρ-invexity, including, of course, its variants such as ρ-pseudoinvexity and ρ-quasiinvexity. Duality results of Wolfe and Mond-Weir [26] type were established. Also, Mond and Smart [14] investigated the duality theory and sufficiency in control problems involving invexity. Later, Bhatia and Mehra [4] studied optimality conditions and duality for multiobjective variational problems with generalized B-invexity. Chandra et al. [5] obtained some optimality conditions and duality results for a class of control problems having a nondifferentiable term in the integrand of the objective functional. Many authors have discussed duality for multiobjective variational problems with different generalized convexities or generalized invexities, such as [12, 16, 17, 27, 28]. Mukherjee and Rao [15] formulated a mixed-type dual for multiobjective variational problems. Several duality theorems have been established relating properly efficient solutions of the primal and dual variational problems under generalized (F, ρ)-convexity (see, for example, Ahmad and Gulati [1]). The notion of (F, α, ρ, d)-type I functions was introduced by Hachimi and Aghezzaf [7]. Thus, they introduced a new class of functions that unified several concepts of generalized type I functions. Some results regarding the efficiency conditions for multi-objective fractional variational problems belong to Mititelu [13], Reddy and Mukherjee [20]. New classes of generalized V-type I invex functions for variational problems have been introduced by Kim and Kim [10] and, later, the sufficiency and duality for multiobjective control problems under generalized (B, ρ)-type I functions were studied by Khazafi et al. [11]. Antczak [2] extended the notions of (ϕ, ρ)-invexity and generalized (ϕ, ρ)-invexity to the continuous case and used these concepts to establish sufficient optimality conditions for the considered class of nonconvex multiobjective variational control problems. In 2021, Treanţă [22] presented the well-posedness of a new class of variational problems with variational inequality constraints, a very useful mathematical tool in the study of optimization problems. Novel approaches to handle the uncertainty in multi-objective optimization problems were presented by Jayswal, Preeti and Treanţă [9] in their book. Many results and approaches

have been put into practice in various branches of operations research. We have mentioned a few of them, but many other people have contributed to the investigation of optimality and/or duality conditions for multiobjective variational control programming problems (see [18, 19, 21]). Treanţă and Calianu [25] investigated a class of multi-objective variational control problems governed by nonconvex simple integral functionals. Recently, Treanţă et al. [24] established the necessary conditions of optimality for new classes of constrained optimization problems involving multiple and curvilinear integral functionals.

Motivated by the previous mentioned papers, in this article, we study and characterize the solution set of a non-convex multi-cost extremization problem. Concretely, we establish some existence results of solutions associated with this optimization model governed by invex, pseudoinvex, or quasiinvex multiple-integral-type functionals. Also, we state a dual connection between the efficient point of the considered non-convex multi-cost extremization problem and the efficient solution of the corresponding dual model. The results of this paper are new in the specialized literature. The main contribution is qiven by the multi-dimensional framework in which the problem is built. More specifically, the presence of multiple integrals as cost functionals in the studied optimization problem. This implies updated definitions for the concept of (χ, ϱ)-invexity (and its variants) associated with functionals governed by multiple integrals and not simple integrals, as usual.

The article is organized as follows: In Section 2 we establish the framework including the notations and basic definitions. Also, we formulate the multi-cost optimization model we are going to study. In section 3, in order to establish some characterization results of solutions associated with the considered problem, first we state the KKT-type efficiency conditions (necessary criterion) associated with a multi-objective optimization problem. Thereafter, we formulate and prove, under various (χ, ϱ)-invexity hypotheses, some theorems which provide the sufficiency of the KKT conditions. In addition, at the end of this section, a dual model is constructed and a relationship between the original extremization problem and the new dual model is established. Last section states the conclusions and some further research directions of the present paper.

2 Preliminaries and notations

For any $h = \left(h^1, h^2, \dots, h^r\right)^T, g = \left(g^1, g^2, \dots, g^r\right)^T$, where $()^T$ represents the transpose, we consider:

(i) $h = g \Longleftrightarrow h^w = g^w$, $w = 1, 2, \ldots, r$;
(ii) $h < g \Longleftrightarrow h^w < g^w$, $w = 1, 2, \ldots, r$;
(iii) $h \leqq g \Longleftrightarrow h^w \leq g^w$, $w = 1, 2, \ldots, r$;
(iv) $h \leq g \Longleftrightarrow h \leqq g$ and $h \neq g$.

Let $\Omega_{v_1,v_2} \subset \mathbb{R}^q$ be a q-dimensional interval and let $H = \{1, 2, \ldots, z\}, E = \{1, 2, \ldots, x\}$ and $A = \{1, 2, \ldots, r\}$ be some index sets. Consider $p(v)$ is a vector-valued (of dimension r) piece-wise smooth function, and $p_\zeta(v) := \frac{\partial p}{\partial v^\zeta}(v)$ represents the partial derivative of $p(v)$, where $v \in \Omega_{v_1,v_2}$. Also, we consider $n(v)$ is a vector-valued (of dimension ω) piece-wise continuous function. Let $B \times C$ be the family of all pairs $(p(v), n(v))$ equipped with the uniform norm $\|p\| = \|p\|_\infty + \|p_\zeta\|_\infty$, and $\|n\| = \|n\|_\infty$, respectively. For notational simplicity, we write $p(v), n(v)$ and $p_\zeta(v)$ as p, n and p_ζ, respectively.

Caristi et al. [6] introduced a generalization of invexity, previously presented by Hanson [8], called (χ, ϱ)-invexity. In the following, we generalize (χ, ϱ)-invexity, stated by Caristi et al. [6] and Antczak [2], for the case of multi-objective control problems. To this aim, we formulate the notion of convexity for $\chi : \Omega_{v_1,v_2} \times (\mathbb{R}^r \times \mathbb{R}^\omega)^2 \times \mathbb{R}^r \times \mathbb{R}^\omega \times \mathbb{R} \to \mathbb{R}$.

Definition 1. The functional $\chi : \Omega_{v_1,v_2} \times (\mathbb{R}^r \times \mathbb{R}^\omega)^2 \times \mathbb{R}^r \times \mathbb{R}^\omega \times \mathbb{R} \to \mathbb{R},\ \chi = \chi(v, p, n, \gamma, \delta; (\cdot, \cdot, \cdot))$ is called *convex on* $\mathbb{R}^{r+\omega+1}$ if, for any $p, \gamma \in \mathbb{R}^r, n, \delta \in \mathbb{R}^\omega$, the following inequality

$$\begin{aligned}&\chi\left(v, p, n, \gamma, \delta; \left(\varepsilon\left(\gamma_1, \tau_1, \varrho_1\right) + (1-\varepsilon)\left(\gamma_2, \tau_2, \varrho_2\right)\right)\right)\\ &\leq \varepsilon\chi\left(v, p, n, \gamma, \delta; \left(\gamma_1, \tau_1, \varrho_1\right)\right) + (1-\varepsilon)\chi\left(v, p, n, \gamma, \delta; \left(\gamma_2, \tau_2, \varrho_2\right)\right)\end{aligned}$$

holds, for all $\gamma_1, \gamma_2 \in \mathbb{R}^r,\ \tau_1, \tau_2 \in \mathbb{R}^\omega,\ \varrho_1, \varrho_2 \in \mathbb{R},\ \varepsilon \in [0, 1],\ v \in \Omega_{v_1,v_2}$.

Consider $\Theta : B \times C \to \mathbb{R}$ is defined as $\Theta(p, n) = \displaystyle\int_{\Omega_{v_1,v_2}} f(v, p, p_\zeta, n) dv$, where $f : \Omega_{v_1,v_2} \times \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R}^\omega \to \mathbb{R}$ is a continuously differentiable multiple integral type functional (see $dv := dv^1 \cdots dv^q$ as volume element in $\mathbb{R}^q$). Next, we establish the (generalized) (χ, ϱ)-invexity associated with Θ.

Definition 2. For a given $(\bar{p}, \bar{n}) \in B \times C$, if there are $\varrho \in \mathbb{R}$ and $\chi : \Omega_{v_1,v_2} \times (\mathbb{R}^r \times \mathbb{R}^\omega)^2 \times \mathbb{R}^r \times \mathbb{R}^\omega \times \mathbb{R} \to \mathbb{R}$, with $\chi = \chi(v, p, n, \bar{p}, \bar{n}; (\cdot, \cdot, \cdot))$ convex on $\mathbb{R}^{r+\omega+1}, \chi(v, p, n, \bar{p}, \bar{n}; (0, 0, \alpha)) \geq 0$ for every $(p, n) \in B \times C$ and $\alpha \in \mathbb{R}_+$, such that

$$\int_{\Omega_{v_1,v_2}} f(v, p, p_\zeta, n) dv - \int_{\Omega_{v_1,v_2}} f(v, \bar{p}, \bar{p}_\zeta, \bar{n}) dv$$

$$\geq [>] \int_{\Omega_{v_1,v_2}} \chi\Big(v, p, n, \bar{p}, \bar{n}; (f_p(v, \bar{p}, \bar{p}_\zeta, \bar{n}) -$$
$$\frac{\partial}{\partial v^\zeta} \left[f_{p_\zeta}(v, \bar{p}, \bar{p}_\zeta, \bar{n})\right], f_n(v, \bar{p}, \bar{p}_\zeta, \bar{n}), \varrho)\Big) dv$$

holds, for all $(p, n) \in B \times C, [(p, n) \neq (\bar{p}, \bar{n})]$, then Θ is called *[strictly] (χ, ϱ)-invex at $(\bar{p}, \bar{n})$ on $B \times C$*. If the above inequality is valid for every $(\bar{p}, \bar{n}) \in B \times C$, then Θ is called *[strictly] (χ, ϱ)-invex on $B \times C$*.

Definition 3. For a given $(\bar{p}, \bar{n}) \in B \times C$, if there are $\varrho \in \mathbb{R}$ and $\chi : \Omega_{v_1,v_2} \times (\mathbb{R}^r \times \mathbb{R}^\omega)^2 \times \mathbb{R}^r \times \mathbb{R}^\omega \times \mathbb{R} \to \mathbb{R}$, with $\chi = \chi(v, p, n, \bar{p}, \bar{n}; (\cdot, \cdot, \cdot))$ convex on $\mathbb{R}^{r+\omega+1}, \chi(v, p, n, \bar{p}, \bar{n}; (0, 0, \alpha)) \geq 0$ for every $(p, n) \in B \times C$ and $\alpha \in \mathbb{R}_+$, such that

$$\int_{\Omega_{v_1,v_2}} f(v, p, p_\zeta, n) dv - \int_{\Omega_{v_1,v_2}} f(v, \bar{p}, \bar{p}_\zeta, \bar{n}) dv$$
$$\leq [<] \int_{\Omega_{v_1,v_2}} \chi\Big(v, p, n, \bar{p}, \bar{n}; (f_p(v, \bar{p}, \bar{p}_\zeta, \bar{n}) -$$
$$\frac{\partial}{\partial v^\zeta} \left[f_{p_\zeta}(v, \bar{p}, \bar{p}_\zeta, \bar{n})\right], f_n(v, \bar{p}, \bar{p}_\zeta, \bar{n}), \varrho)\Big) dv$$

holds, for all $(p, n) \in B \times C, [(p, n) \neq (\bar{p}, \bar{n})]$, then Θ is called *[strictly] (χ, ϱ)-incave at $(\bar{p}, \bar{n})$ on $B \times C$*. If the inequality given above is satisfied for every $(\bar{p}, \bar{n}) \in B \times C$, then Θ is called *[strictly] (χ, ϱ)-incave on $B \times C$*.

Definition 4. For a given $(\bar{p}, \bar{n}) \in B \times C$, if there are $\varrho \in \mathbb{R}$ and $\chi : \Omega_{v_1,v_2} \times (\mathbb{R}^r \times \mathbb{R}^\omega)^2 \times \mathbb{R}^r \times \mathbb{R}^\omega \times \mathbb{R} \to \mathbb{R}$, with $\chi = \chi(v, p, n, \bar{p}, \bar{n}; (\cdot, \cdot, \cdot))$ convex on $\mathbb{R}^{r+\omega+1}, \chi(v, p, n, \bar{p}, \bar{n}; (0, 0, \alpha)) \geq 0$ for every $(p, n) \in B \times C$ and $\alpha \in \mathbb{R}_+$, such that

$$\int_{\Omega_{v_1,v_2}} f(v, p, p_\zeta, n) dv < \int_{\Omega_{v_1,v_2}} f(v, \bar{p}, \bar{p}_\zeta, \bar{n}) dv$$

implies

$$\int_{\Omega_{v_1,v_2}} \chi\Big(v, p, n, \bar{p}, \bar{n}; (f_p(v, \bar{p}, \bar{p}_\zeta, \bar{n}) -$$
$$\frac{\partial}{\partial v^\zeta} \left[f_{p_\zeta}(v, \bar{p}, \bar{p}_\zeta, \bar{n})\right], f_n(v, \bar{p}, \bar{p}_\zeta, \bar{n}), \varrho)\Big) dv < 0$$

holds, for all $(p, n) \in B \times C$, then Θ is called *(χ, ϱ)-pseudoinvex at $(\bar{p}, \bar{n})$ on $B \times C$*. If the relation given above is satisfied for every $(\bar{p}, \bar{n}) \in B \times C$, then Θ is called *(χ, ϱ)-pseudoinvex on $B \times C$*.

Definition 5. For a given $(\bar{p}, \bar{n}) \in B \times C$, if there are $\varrho \in \mathbb{R}$ and $\chi : \Omega_{v_1,v_2} \times (\mathbb{R}^r \times \mathbb{R}^\omega)^2 \times \mathbb{R}^r \times \mathbb{R}^\omega \times \mathbb{R} \to \mathbb{R}$, with $\chi = \chi(v, p, n, \bar{p}, \bar{n}; (\cdot, \cdot, \cdot))$ convex on $\mathbb{R}^{r+\omega+1}$, $\chi(v, p, n, \bar{p}, \bar{n}; (0, 0, \alpha)) \geq 0$ for every $(p, n) \in B \times C$ and $\alpha \in \mathbb{R}_+$, such that

$$\int_{\Omega_{v_1,v_2}} f(v, p, p_\zeta, n) dv \leq \int_{\Omega_{v_1,v_2}} f(v, \bar{p}, \bar{p}_\zeta, \bar{n}) dv$$

implies

$$\int_{\Omega_{v_1,v_2}} \chi\Big(v, p, n, \bar{p}, \bar{n}; (f_p(v, \bar{p}, \bar{p}_\zeta, \bar{n}) - \frac{\partial}{\partial v^\zeta}\left[f_{p_\zeta}(v, \bar{p}, \bar{p}_\zeta, \bar{n})\right], f_n(v, \bar{p}, \bar{p}_\zeta, \bar{n}), \varrho)\Big) dv < 0$$

holds, for all $(p, n) \in B \times C, (p, n) \neq (\bar{p}, \bar{n})$, then Θ is called *strictly* (χ, ϱ)-*pseudoinvex at* $(\bar{p}, \bar{n})$ *on* $B \times C$. If the relation given above is satisfied for every $(\bar{p}, \bar{n}) \in B \times C$, then Θ is called *strictly* (χ, ϱ)-*pseudoinvex on* $B \times C$.

Definition 6. For a given $(\bar{p}, \bar{n}) \in B \times C$, if there are $\varrho \in \mathbb{R}$ and $\chi : \Omega_{v_1,v_2} \times (\mathbb{R}^r \times \mathbb{R}^\omega)^2 \times \mathbb{R}^r \times \mathbb{R}^\omega \times \mathbb{R} \to \mathbb{R}$, with $\chi = \chi(v, p, n, \bar{p}, \bar{n}; (\cdot, \cdot, \cdot))$ convex on $\mathbb{R}^{r+\omega+1}$, $\chi(v, p, n, \bar{p}, \bar{n}; (0, 0, \alpha)) \geq 0$ for every $(p, n) \in B \times C$ and $\alpha \in \mathbb{R}_+$, such that

$$\int_{\Omega_{v_1,v_2}} f(v, p, p_\zeta, n) dv \leq \int_{\Omega_{v_1,v_2}} f(v, \bar{p}, \bar{p}_\zeta, \bar{n}) dv$$

implies

$$\int_{\Omega_{v_1,v_2}} \chi\Big(v, p, n, \bar{p}, \bar{n}; (f_p(v, \bar{p}, \bar{p}_\zeta, \bar{n}) - \frac{\partial}{\partial v^\zeta}\left[f_{p_\zeta}(v, \bar{p}, \bar{p}_\zeta, \bar{n})\right], f_n(v, \bar{p}, \bar{p}_\zeta, \bar{n}), \varrho)\Big) dv \leq 0$$

holds, for all $(p, n) \in B \times C$, then Θ is called (χ, ϱ)-*quasiinvex at* $(\bar{p}, \bar{n})$ *on* $B \times C$. If the relation given above is satisfied for every $(\bar{p}, \bar{n}) \in B \times C$, then Θ is called (χ, ϱ)-*quasiinvex on* $B \times C$.

The notion of (χ, ϱ)-invexity generalizes many types of convexities. In this regard, a functional Θ can be (χ, ϱ)-invex but not invex.

This paper focuses on efficiency criteria and the associated duals for the following nonconvex multi-cost extremization problem:

$$\text{(NMEP)} \quad \min_{(p,n)} \int_{\Omega_{v_1,v_2}} \Upsilon(v, p(v), n(v)) dv$$

$$= \left(\int_{\Omega_{v_1,v_2}} \Upsilon^1(v,p(v),n(v))dv, \ldots, \int_{\Omega_{v_1,v_2}} \Upsilon^z(v,p(v),n(v))dv \right)$$

$$\begin{aligned} \text{subject to} \quad & U(v,p(v),n(v)) \leqq 0, \quad v \in \Omega_{v_1,v_2}, \\ & Z(v,p(v),n(v)) = p_\zeta(v), \quad v \in \Omega_{v_1,v_2}, \\ & p(v_1) = p_1 = \text{given}, \quad p(v_2) = p_2 = \text{given}, \end{aligned}$$

where $\Upsilon = \left(\Upsilon^1, \ldots, \Upsilon^z\right) : \Omega_{v_1,v_2} \times \mathbb{R}^r \times \mathbb{R}^\omega \to \mathbb{R}^z$ is a z-dimensional C^1-class functional, and the constraint functionals $U : \Omega_{v_1,v_2} \times \mathbb{R}^r \times \mathbb{R}^\omega \to \mathbb{R}^x$ and $Z : \Omega_{v_1,v_2} \times \mathbb{R}^r \times \mathbb{R}^\omega \to \mathbb{R}^r$ are assumed to be continuously differentiable x-dimensional and r-dimensional functionals, respectively.

Let V be the feasible solution set for (NMEP), that is

$$V = \{(p,n) \in B \times C \text{ satisfying the mentioned constraints in (NMEP)}\}.$$

Definition 7. A pair $(\bar{p},\bar{n}) \in V$ associated to (NMEP) is an *efficient point* of (NMEP) if there exists no other $(p,n) \in V$ such that

$$\int_{\Omega_{v_1,v_2}} \Upsilon(v,p,n)dv \leq \int_{\Omega_{v_1,v_2}} \Upsilon(v,\bar{p},\bar{n})dv$$

that is, there exists no other $(p,n) \in V$ such that

$$\begin{aligned} &\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,p,n)dv \leq \int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,\bar{p},\bar{n})dv, \quad \forall \iota \in H, \\ &\int_{\Omega_{v_1,v_2}} \Upsilon^s(v,p,n)dv < \int_{\Omega_{v_1,v_2}} \Upsilon^s(v,\bar{p},\bar{n})dv, \quad \text{for some } s \in H. \end{aligned}$$

3 Main results

In the following, to establish some characterization results of solutions associated with (NMEP), we state the KKT conditions (necessary criteria of efficiency) associated with a multi-objective optimization problem (see Treanţă [23]).

Theorem 1. *Let $(\bar{p},\bar{n})$ be a normal efficient point for (NMEP) and the KKT constraint qualification be satisfied. Then there are $\bar{\eta} \in \mathbb{R}^z$, $\bar{\lambda}(\cdot) : \Omega_{v_1,v_2} \to \mathbb{R}^x$ and $\bar{\pi}(\cdot) : \Omega_{v_1,v_2} \to \mathbb{R}^r$ satisfying*

$$\bar{\eta}^T \Upsilon_p(v,\bar{p},\bar{n}) + \bar{\lambda}(v)^T G_p(v,\bar{p},\bar{n}) + \bar{\pi}(v)^T H_p(v,\bar{p},\bar{p}_\zeta,\bar{n})$$

$$= \frac{\partial}{\partial v^{\zeta}} \left[\bar{\eta}^T \Upsilon_{p_\zeta}(v,\bar{p},\bar{n}) + \bar{\lambda}(v)^T G_{p_\zeta}(v,\bar{p},\bar{n}) + \bar{\pi}(v)^T H_{p_\zeta}(v,\bar{p},\bar{p}_\zeta,\bar{n}) \right], \quad (1)$$

$$\bar{\eta}^T \Upsilon_n(v,\bar{p},\bar{n}) + \bar{\lambda}(v)^T G_n(v,\bar{p},\bar{n}) + \bar{\pi}(v)^T H_n(v,\bar{p},\bar{p}_\zeta,\bar{n}) = 0,\ v \in \Omega_{v_1,v_2}, \quad (2)$$

$$\int_{\Omega_{v_1,v_2}} \bar{\lambda}(v)^T U(v,\bar{p},\bar{n}) dv = 0,\ \bar{\eta} \geq 0,\ \bar{\eta}^T e = 1,\ \bar{\lambda}(v) \geqq 0,\ v \in \Omega_{v_1,v_2}, \quad (3)$$

except at discontinuities, with $S := Z - p_\zeta$.

For simplicity, we consider λ for $\lambda(v)$ and π for $\pi(v)$.

Theorem 2. *If $(\bar{p},\bar{n}) \in V$, the KKT criteria in (1)-(3) are fulfilled at this pair, with $\bar{\eta} \in \mathbb{R}^z$ and $\bar{\lambda}(\cdot) : \Omega_{v_1,v_2} \to \mathbb{R}^x$ and $\bar{\pi}(\cdot) : \Omega_{v_1,v_2} \to \mathbb{R}^r$, and the hypotheses are valid:*

(a) $\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,p,n)dv, \iota \in H$, is strictly $(\chi, \varrho_{\Upsilon^\iota})$-invex at $(\bar{p},\bar{n})$ on V;

(n) $\int_{\Omega_{v_1,v_2}} U^l(v,p,n)dv, l \in E$, is (χ, ϱ_{U^l})-invex at $(\bar{p},\bar{n})$ on V;

(c) $\int_{\Omega_{v_1,v_2}} S^j(v,p,p_\zeta,n)dv, j \in A^+(v) = \{j \in A : \bar{\pi}_j(v) > 0\}$, is (χ, ϱ_{S^j})-invex at $(\bar{p},\bar{n})$ on V;

(d) $-\int_{\Omega_{v_1,v_2}} S^j(v,p,p_\zeta,n)dv, j \in A^-(v) = \{j \in A : \bar{\pi}_j(v) < 0\}$, is (χ, ϱ_{S^j})-invex at $(\bar{p},\bar{n})$ on V;

(e) $\sum_{\iota=1}^{z} \bar{\eta}_\iota \varrho_{\Upsilon^\iota} + \sum_{l=1}^{x} \bar{\lambda}_l \varrho_{U^l} + \sum_{j \in A^+(v)} \bar{\pi}_j \varrho_{S^j} - \sum_{j \in A^-(v)} \bar{\pi}_j \varrho_{S^j} \geq 0$,

then $(\bar{p},\bar{n})$ is an efficient point of (NMEP).

Proof. Suppose, contrary to the result, that $(\bar{p},\bar{n})$ is not an efficient point of (NMEP). Then, there exists $(\tilde{p},\tilde{n}) \in V$ such that

$$\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,\tilde{p},\tilde{n})dv \leq \int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,\bar{p},\bar{n})dv, \quad \forall \iota \in H \quad (4)$$

and

$$\int_{\Omega_{v_1,v_2}} \Upsilon^s(v,\tilde{p},\tilde{n})dv < \int_{\Omega_{v_1,v_2}} \Upsilon^s(v,\bar{p},\bar{n})dv, \text{ for some } s \in H. \quad (5)$$

Due to (a)-(d), the inequalities are valid

$$\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,\tilde{p},\tilde{n})dv - \int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,\bar{p},\bar{n})dv$$

$$> \int_{\Omega_{v_1,v_2}} \chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(\Upsilon^{\iota}_p(v, \bar{p}, \bar{n}) -$$

$$\frac{\partial}{\partial v^{\zeta}}\left[\Upsilon^{\iota}_{p_{\zeta}}(v, \bar{p}, \bar{n})\right], \Upsilon^{\iota}_n(v, \bar{p}, \bar{n}), \varrho_{\Upsilon^{\iota}}\Big)\Big)dv, \quad \iota \in H, \tag{6}$$

$$\int_{\Omega_{v_1,v_2}} U^l(v, \tilde{p}, \tilde{n})dv - \int_{\Omega_{v_1,v_2}} U^l(v, \bar{p}, \bar{n})dv$$

$$\geq \int_{\Omega_{v_1,v_2}} \chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(G^l_p(v, \bar{p}, \bar{n}) -$$

$$\frac{\partial}{\partial v^{\zeta}}\left[G^l_{p_{\zeta}}(v, \bar{p}, \bar{n})\right], G^l_n(v, \bar{p}, \bar{n}), \varrho_{U^l}\Big)\Big)dv, \quad l \in E, \tag{7}$$

$$\int_{\Omega_{v_1,v_2}} S^j(v, \tilde{p}, \dot{\tilde{p}}, \tilde{n})dv - \int_{\Omega_{v_1,v_2}} S^j(v, \bar{p}, \bar{p}_{\zeta}, \bar{n})dv$$

$$\geq \int_{\Omega_{v_1,v_2}} \chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(H^j_p(v, \bar{p}, \bar{p}_{\zeta}, \bar{n})$$

$$- \frac{\partial}{\partial v^{\zeta}}\left[H^j_{p_{\zeta}}(v, \bar{p}, \bar{p}_{\zeta}, \bar{n})\right], H^j_n(v, \bar{p}, \bar{p}_{\zeta}, \bar{n}), \varrho_{S^j}\Big)\Big)dv, \ j \in A^+(v), \tag{8}$$

$$- \int_{\Omega_{v_1,v_2}} S^j(v, \tilde{p}, \dot{\tilde{p}}, \tilde{n})dv + \int_{\Omega_{v_1,v_2}} S^j(v, \bar{p}, \bar{p}_{\zeta}, \bar{n})dv$$

$$\geq \int_{\Omega_{v_1,v_2}} \chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(- H^j_p(v, \bar{p}, \bar{p}_{\zeta}, \bar{n}) - \frac{\partial}{\partial v^{\zeta}}\left[-H^j_{p_{\zeta}}(v, \bar{p}, \bar{p}_{\zeta}, \bar{n})\right],$$

$$- H^j_n(v, \bar{p}, \bar{p}_{\zeta}, \bar{n}), \varrho_{S^j}\Big)\Big)dv, \ j \in A^-(v). \tag{9}$$

By using (4)-(6) and $\bar{\eta} \geq 0$, we obtain

$$\int_{\Omega_{v_1,v_2}} \bar{\eta}_{\iota}\chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(\Upsilon^{\iota}_p(v, \bar{p}, \bar{n})$$

$$- \frac{\partial}{\partial v^{\zeta}}\left[\Upsilon^{\iota}_{p_{\zeta}}(v, \bar{p}, \bar{n})\right], \Upsilon^{\iota}_n(v, \bar{p}, \bar{n}), \varrho_{\Upsilon^{\iota}}\Big)\Big)dv \leq 0, \tag{10}$$

for $\iota \in H$, and

$$\int_{\Omega_{v_1,v_2}} \bar{\eta}_s\chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(\Upsilon^s_p(v, \bar{p}, \bar{n})$$

$$- \frac{\partial}{\partial v^{\zeta}}\left[\Upsilon^s_{p_{\zeta}}(v, \bar{p}, \bar{n})\right], \Upsilon^s_n(v, \bar{p}, \bar{n}), \varrho_{\Upsilon^s}\Big)\Big)dv < 0, \tag{11}$$

for at least $s \in H$. By adding (10) and (11), it follows

$$\int_{\Omega_{v_1,v_2}} \sum_{\iota=1}^{z} \bar{\eta}_\iota \chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(\Upsilon_p^\iota(v, \bar{p}, \bar{n})$$

$$-\frac{\partial}{\partial v^\zeta}\left[\Upsilon_{p_\zeta}^\iota(v, \bar{p}, \bar{n})\right], \Upsilon_n^\iota(v, \bar{p}, \bar{n}), \varrho_{\Upsilon^\iota}\Big)\Big) dv < 0. \tag{12}$$

Since $\bar{\lambda}_l(v) \geq 0, l \in E$, the relation (7) provides

$$\int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \bar{\lambda}_l U^l(v, \tilde{p}, \tilde{n}) dv - \int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \bar{\lambda}_l U^l(v, \bar{p}, \bar{n}) dv$$

$$\geq \int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \bar{\lambda}_l \chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(G_p^l(v, \bar{p}, \bar{n})$$

$$-\frac{\partial}{\partial v^\zeta}\left[G_{p_\zeta}^l(v, \bar{p}, \bar{n})\right], G_n^l(v, \bar{p}, \bar{n}), \varrho_{U^l}\Big)\Big) dv. \tag{13}$$

Considering the feasibility of the pair $(\tilde{p}, \tilde{n})$ in (NMEP) together with the condition given in (3), we get

$$\int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \bar{\lambda}_l \chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(G_p^l(v, \bar{p}, \bar{n})$$

$$-\frac{\partial}{\partial v^\zeta}\left[G_{p_\zeta}^l(v, \bar{p}, \bar{n})\right], G_n^l(v, \bar{p}, \bar{n}), \varrho_{U^l}\Big)\Big) dv \leq 0. \tag{14}$$

The relations given in (8) and (9) produce, respectively,

$$\int_{\Omega_{v_1,v_2}} \sum_{j \in A^+(v)} \bar{\pi}_j S^j(v, \tilde{p}, \dot{\tilde{p}}, \tilde{n}) dv - \int_{\Omega_{v_1,v_2}} \sum_{j \in A^+(v)} \bar{\pi}_j S^j(v, \bar{p}, \bar{p}_\zeta, \bar{n}) dv$$

$$\geq \int_{\Omega_{v_1,v_2}} \sum_{j \in A^+(v)} \bar{\pi}_j \chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(H_p^j(v, \bar{p}, \bar{p}_\zeta, \bar{n}) - \frac{\partial}{\partial v^\zeta}\left[H_{p_\zeta}^j(v, \bar{p}, \bar{p}_\zeta, \bar{n})\right],$$

$$H_n^j(v, \bar{p}, \bar{p}_\zeta, \bar{n}), \varrho_{S^j}\Big)\Big) dv, \tag{15}$$

$$\int_{\Omega_{v_1,v_2}} \sum_{j \in A^-(v)} \bar{\pi}_j S^j(v, \tilde{p}, \dot{\tilde{p}}, \tilde{n}) dv - \int_{\Omega_{v_1,v_2}} \sum_{j \in A^-(v)} \bar{\pi}_j S^j(v, \bar{p}, \bar{p}_\zeta, \bar{n}) dv$$

$$\geq \int_{\Omega_{v_1,v_2}} \sum_{j \in A^-(v)} (-\bar{\pi}_j)\, \chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(-H_p^j(v, \bar{p}, \bar{p}_\zeta, \bar{n})$$

$$-\frac{\partial}{\partial v^\zeta}\left[-H_{p_\zeta}^j(v, \bar{p}, \bar{p}_\zeta, \bar{n})\right], -H_n^j(v, \bar{p}, \bar{p}_\zeta, \bar{n}), \varrho_{S^j}\Big)\Big) dv. \tag{16}$$

By adding (15) and (16), it results

$$\int_{\Omega_{v_1,v_2}} \sum_{j\in A} \bar{\pi}_j S^j(v,\tilde{p},\dot{\tilde{p}},\tilde{n})dv - \int_{\Omega_{v_1,v_2}} \sum_{j\in A} \bar{\pi}_j S^j(v,\bar{p},\bar{p}_\zeta,\bar{n})dv$$
$$\geq \int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \bar{\pi}_j \chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n}) - \frac{\partial}{\partial v^\zeta}\left[H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\right],$$
$$H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv$$
$$+\int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} (-\bar{\pi}_j)\,\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(-H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n})$$
$$-\frac{\partial}{\partial v^\zeta}\left[-H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\right],-H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv.$$

By considering the feasibility of the pair $(\tilde{p},\tilde{n})$ in (NMEP), together with (2) and (3), we have

$$\int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \bar{\pi}_j \chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n}) - \frac{\partial}{\partial v^\zeta}\left[H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\right],$$
$$H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv$$
$$+\int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} (-\bar{\pi}_j)\,\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(-H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n})$$
$$-\frac{\partial}{\partial v^\zeta}\left[-H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\right],-H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv \leq 0. \tag{17}$$

By relations (12), (14) and (17), we get

$$\int_{\Omega_{v_1,v_2}} \sum_{\iota=1}^{z} \bar{\eta}_\iota \chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(\Upsilon_p^\iota(v,\bar{p},\bar{\bar{p}})$$
$$-\frac{\partial}{\partial v^\zeta}\left[\Upsilon_{p_\zeta}^\iota(v,\bar{p},\bar{n})\right],\Upsilon_n^\iota(v,\bar{p},\bar{n}),\varrho_{\Upsilon^\iota}\Big)\Big)dv$$
$$+\int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \bar{\lambda}_l \chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(G_p^l(v,\bar{p},\bar{n})$$
$$-\frac{\partial}{\partial v^\zeta}\left[G_{p_\zeta}^l(v,\bar{p},\bar{n})\right],G_n^l(v,\bar{p},\bar{n}),\varrho_{U^l}\Big)\Big)dv$$
$$+\int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \bar{\pi}_j \chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n}) - \frac{\partial}{\partial v^\zeta}\left[H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\right],$$
$$H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv$$

$$+\int_{\Omega_{v_1,v_2}}\sum_{j\in A^-(v)}(-\bar{\pi}_j)\,\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(-H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n})$$
$$-\frac{\partial}{\partial v^\zeta}\Big[-H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\Big],-H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv<0. \tag{18}$$

Let us denote

$$\widehat{\eta}_\iota=\frac{\bar{\eta}_\iota}{\sum_{\iota=1}^z\bar{\eta}_\iota+\sum_{l=1}^x\bar{\lambda}_l(v)+\sum_{j\in A^+(v)}\bar{\pi}_j(v)-\sum_{j\in A^-(v)}\bar{\pi}_j(v)},$$
$$\iota\in H, \tag{19}$$
$$\widehat{\lambda}_l(v)=\frac{\bar{\lambda}_l(v)}{\sum_{\iota=1}^z\bar{\eta}_\iota+\sum_{l=1}^x\bar{\lambda}_l(v)+\sum_{j\in A^+(v)}\bar{\pi}_j(v)-\sum_{j\in A^-(v)}\bar{\pi}_j(v)},$$
$$l\in E, \tag{20}$$
$$\widehat{\pi}_j(v)=\frac{\bar{\pi}_j(v)}{\sum_{\iota=1}^z\bar{\eta}_\iota+\sum_{l=1}^x\bar{\lambda}_l(v)+\sum_{j\in A^+(v)}\bar{\pi}_j(v)-\sum_{j\in A^-(v)}\bar{\pi}_j(v)},$$
$$j\in A^+(v), \tag{21}$$
$$\widehat{\pi}_j(v)=\frac{-\bar{\pi}_j(v)}{\sum_{\iota=1}^z\bar{\eta}_\iota+\sum_{l=1}^x\bar{\lambda}_l(v)+\sum_{j\in A^+(v)}\bar{\pi}_j(v)-\sum_{j\in A^-(v)}\bar{\pi}_j(v)},$$
$$j\in A^-(v). \tag{22}$$

By (19)-(22), we get that $0\le\widehat{\eta}_\iota\le 1,\iota\in H$, but $\widehat{\eta}_\iota>0$ for at least $\iota\in H, 0\le \widehat{\lambda}_l(v)\le 1, l\in E, 0\le\widehat{\pi}_j(v)\le 1, j\in A$, and, in addition,

$$\sum_{\iota=1}^z\widehat{\eta}_\iota+\sum_{l=1}^x\widehat{\lambda}_l(v)+\sum_{j\in A^+(v)}\widehat{\pi}_j(v)+\sum_{j\in A^-(v)}\widehat{\pi}_j(v)=1. \tag{23}$$

Combining (18)-(22), we get

$$\int_{\Omega_{v_1,v_2}}\sum_{\iota=1}^z\widehat{\eta}_\iota\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(\Upsilon_p^\iota(v,\bar{p},\bar{n})$$
$$-\frac{\partial}{\partial v^\zeta}\Big[\Upsilon_{p_\zeta}^\iota(v,\bar{p},\bar{n})\Big],\Upsilon_n^\iota(v,\bar{p},\bar{n}),\varrho_{\Upsilon^\iota}\Big)\Big)dv$$
$$+\int_{\Omega_{v_1,v_2}}\sum_{l=1}^x\widehat{\lambda}_l\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(G_p^l(v,\bar{p},\bar{n})$$
$$-\frac{\partial}{\partial v^\zeta}\Big[G_{p_\zeta}^l(v,\bar{p},\bar{n})\Big],G_n^l(v,\bar{p},\bar{n}),\varrho_{U^l}\Big)\Big)dv$$

$$+\int_{\Omega_{v_1,v_2}}\sum_{j\in A^+(v)}\widehat{\pi}_j\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n})-\frac{\partial}{\partial v^\zeta}\Big[H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\Big],$$
$$H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv$$
$$+\int_{\Omega_{v_1,v_2}}\sum_{j\in A^-(v)}\widehat{\pi}_j\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(-H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n})-\frac{\partial}{\partial v^\zeta}\Big[-H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\Big],$$
$$-H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv<0. \tag{24}$$

By Definition 2, we get $\chi(v,\tilde{p},\tilde{n},\bar{p},\bar{n};(\cdot,\cdot,\cdot))$ is convex on $\mathbb{R}^{r+\omega+1}$. Since (23) is true, then by (24) and Definition 1, we get

$$\int_{\Omega_{v_1,v_2}}\chi\Bigg(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Bigg(\Bigg[\sum_{\iota=1}^{z}\widehat{\eta}_\iota\Upsilon_p^\iota(v,\bar{p},\bar{n})+\sum_{l=1}^{x}\widehat{\lambda}_lG_p^l(v,\bar{p},\bar{n})$$
$$+\sum_{j\in A^+(v)}\widehat{\pi}_jH_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n})+\sum_{j\in A^-(v)}\widehat{\pi}_jH_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\Bigg]$$
$$-\frac{\partial}{\partial v^\zeta}\Bigg[\sum_{\iota=1}^{z}\widehat{\eta}_\iota\Upsilon_{p_\zeta}^\iota(v,\bar{p},\bar{n})+\sum_{l=1}^{x}\widehat{\lambda}_lG_{p_\zeta}^l(v,\bar{p},\bar{n})$$
$$+\sum_{j\in A^+(v)}\widehat{\pi}_jH_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})+\sum_{j\in A^-(v)}\widehat{\pi}_jH_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\Bigg],$$
$$\sum_{\iota=1}^{z}\widehat{\eta}_\iota\Upsilon_n^\iota(v,\bar{p},\bar{n})+\sum_{l=1}^{x}\widehat{\lambda}_lG_n^l(v,\bar{p},\bar{n})$$
$$+\sum_{j\in A^+(v)}\widehat{\pi}_jH_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n})+\sum_{j\in A^-(v)}\widehat{\pi}_jH_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),$$
$$\sum_{\iota=1}^{z}\widehat{\eta}_\iota\varrho_{\Upsilon^\iota}+\sum_{l=1}^{x}\widehat{\lambda}_l\varrho_{U^l}+\sum_{j\in A^+(v)\cup A^-(v)}\widehat{\pi}_j\varrho_{S^j}\Bigg)\Bigg)dv<0.$$

Hence, the KKT conditions yield

$$\int_{\Omega_{v_1,v_2}}\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(0,0,\sum_{\iota=1}^{z}\widehat{\eta}_\iota\varrho_{\Upsilon^\iota}+\sum_{l=1}^{x}\widehat{\lambda}_l\varrho_{U^l}+\sum_{j\in A^+(v)\cup A^-(v)}\widehat{\pi}_j\varrho_{S^j}\Big)\Big)dv<0. \tag{25}$$

From the hypothesis (e), we have

$$\sum_{\iota=1}^{z} \widehat{\eta}_\iota \varrho_{\Upsilon^\iota} + \sum_{l=1}^{x} \widehat{\lambda}_l \varrho_{U^l} + \sum_{j \in A^+(v) \cup A^-(v)} \widehat{\pi}_j \varrho_{S^j} \geq 0. \tag{26}$$

By Definition 2, it follows that $\chi(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; (0, 0, \alpha)) \geq 0$ for any $\alpha \in \mathbb{R}_+$. Thus, realtion (26) implies that

$$\int_{\Omega_{v_1,v_2}} \chi\Big(v, \tilde{p}, \tilde{n}, \bar{p}, \bar{n}; \Big(0, 0, \sum_{\iota=1}^{z} \widehat{\eta}_\iota \varrho_{\Upsilon^\iota} + \sum_{l=1}^{x} \widehat{\lambda}_l \varrho_{U^l} + \sum_{j \in A^+(v) \cup A^-(v)} \widehat{\pi}_j \varrho_{S^j}\Big)\Big) dv \geq 0$$

is valid, contradicting (25). In consequence, the proof is complete. □

Theorem 3. *If $(\bar{p}, \bar{n}) \in V$, the KKT criteria in (1)-(3) are satisfied at this pair, with $\bar{\eta} \in \mathbb{R}^z$ and $\bar{\lambda}(\cdot) : \Omega_{v_1,v_2} \to \mathbb{R}^x$ and $\bar{\pi}(\cdot) : \Omega_{v_1,v_2} \to \mathbb{R}^r$, and the statements are fulfilled:*

(a) $\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v, p, n) dv, \iota \in H$, is strictly $(\chi, \varrho_{\Upsilon^\iota})$-pseudoinvex at $(\bar{p}, \bar{n})$ on V;

(n) $\int_{\Omega_{v_1,v_2}} U^l(v, p, n) dv, l \in E$, is (χ, ϱ_{U^l})-quasiinvex at $(\bar{p}, \bar{n})$ on V;

(c) $\int_{\Omega_{v_1,v_2}} S^j(v, p, p_\zeta, n) dv, j \in A^+(v)$, is (χ, ϱ_{S^j})-quasiinvex at $(\bar{p}, \bar{n})$ on V;

(d) $-\int_{\Omega_{v_1,v_2}} S^j(v, p, p_\zeta, n) dv, j \in A^-(v)$, is (χ, ϱ_{S^j})-quasiinvex at $(\bar{p}, \bar{n})$ on V;

(e) $\sum_{\iota=1}^{z} \bar{\eta}_\iota \varrho_{\Upsilon^\iota} + \sum_{l=1}^{x} \bar{\lambda}_\iota \varrho_{U^l} + \sum_{j \in A^+(v)} \bar{\pi}_j \varrho_{S^j} - \sum_{j \in A^-(v)} \bar{\pi}_j \varrho_{S^j} \geq 0$,

then $(\bar{p}, \bar{n})$ is an efficient point of (NMEP).

Proof. Suppose, contrary to the result, that $(\bar{p}, \bar{n})$ is not an efficient point of (NMEP). Then, there exists $(\tilde{p}, \tilde{n}) \in V$ such that

$$\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v, \tilde{p}, \tilde{n}) dv \leq \int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v, \bar{p}, \bar{n}) dv, \quad \iota \in H \tag{27}$$

and

$$\int_{\Omega_{v_1,v_2}} \Upsilon^s(v, \tilde{p}, \tilde{n}) dv < \int_{\Omega_{v_1,v_2}} \Upsilon^s(v, \bar{p}, \bar{n}) dv, \quad \text{for some } s \in H. \tag{28}$$

By using Definition 5, relations given in (27) and (28) yield

$$\int_{\Omega_{v_1,v_2}} \chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(\Upsilon^{\iota}_{p}(v,\bar{p},\bar{n})$$

$$-\frac{\partial}{\partial v^{\zeta}}\left[\Upsilon^{\iota}_{p_{\zeta}}(v,\bar{p},\bar{n})\right],\Upsilon^{\iota}_{n}(v,\bar{p},\bar{n}),\varrho_{\Upsilon^{\iota}}\Big)\Big)dv<0,\ \iota\in H,$$

and, since $\bar{\eta}\geq 0$, then the above inequality gives

$$\int_{\Omega_{v_1,v_2}} \sum_{\iota=1}^{z}\bar{\eta}_{\iota}\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(\Upsilon^{\iota}_{p}(v,\bar{p},\bar{n})$$

$$-\frac{\partial}{\partial v^{\zeta}}\left[\Upsilon^{\iota}_{p_{\zeta}}(v,\bar{p},\bar{n})\right],\Upsilon^{\iota}_{n}(v,\bar{p},\bar{n}),\varrho_{\Upsilon^{\iota}}\Big)\Big)dv<0. \tag{29}$$

By using the feasibility of $(\bar{p},\bar{n})$ and $(\tilde{p},\tilde{n})$ in (NMEP), together with the KKT necessary efficiency criteria, it follows

$$\int_{\Omega_{v_1,v_2}} \bar{\lambda}_l U^l(v,\tilde{p},\tilde{n})dv\leq\int_{\Omega_{v_1,v_2}} \bar{\lambda}_l U^l(v,\bar{p},\bar{n})dv,\quad l\in E.$$

By using Definition 6, the assumption in (n) implies

$$\int_{\Omega_{v_1,v_2}} \bar{\lambda}_l\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(G^{l}_{p}(v,\bar{p},\bar{n})$$

$$-\frac{\partial}{\partial v^{\zeta}}\left[G^{l}_{p_{\zeta}}(v,\bar{p},\bar{n})\right],G^{l}_{n}(v,\bar{p},\bar{n}),\varrho_{U^{l}}\Big)\Big)dv\leq 0,\ l\in E.$$

Adding the inequalities above, we obtain

$$\int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x}\bar{\lambda}_l\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(G^{l}_{p}(v,\bar{p},\bar{n})$$

$$-\frac{\partial}{\partial v^{\zeta}}\left[G^{l}_{p_{\zeta}}(v,\bar{p},\bar{n})\right],G^{l}_{n}(v,\bar{p},\bar{n}),\varrho_{U^{l}}\Big)\Big)dv\leq 0. \tag{30}$$

Further, by using the feasibility of $(\bar{p},\bar{n})$ and $(\tilde{p},\tilde{n})$ in (NMEP), we have

$$\int_{\Omega_{v_1,v_2}} S^j(v,\tilde{p},\dot{\tilde{p}},\tilde{n})dv=\int_{\Omega_{v_1,v_2}} S^j(v,\bar{p},\bar{p}_{\zeta},\bar{n})dv,\ j\in A^{+}(v), \tag{31}$$

$$\int_{\Omega_{v_1,v_2}} -S^j(v,\tilde{p},\dot{\tilde{p}},\tilde{n})dv=\int_{\Omega_{v_1,v_2}} -S^j(v,\bar{p},\bar{p}_{\zeta},\bar{n})dv,\ j\in A^{-}(v). \tag{32}$$

Thus, by (c) and (d), the relations given in (31) and (32) imply

$$\int_{\Omega_{v_1,v_2}} \chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n}) - \frac{\partial}{\partial v^\zeta}\left[H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\right],$$

$$H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv \leq 0,\ j \in A^+(v), \tag{33}$$

$$\int_{\Omega_{v_1,v_2}} \chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(-H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n}) - \frac{\partial}{\partial v^\zeta}\left[-H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\right],$$

$$-H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv \leq 0,\ j \in A^-(v). \tag{34}$$

Thus,

$$\int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \bar{\pi}_j\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n}) - \frac{\partial}{\partial v^\zeta}\left[H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\right],$$

$$H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv \leq 0, \tag{35}$$

$$\int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} (-\bar{\pi}_j)\,\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(-H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n}) - \frac{\partial}{\partial v^\zeta}\left[-H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\right],$$

$$-H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv \leq 0. \tag{36}$$

By using relations in (29), (30), (35) and (36), it follows

$$\int_{\Omega_{v_1,v_2}} \sum_{\iota=1}^{z} \bar{\eta}_\iota\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(\Upsilon_p^\iota(v,\bar{p},\bar{n})$$

$$-\frac{\partial}{\partial v^\zeta}\left[\Upsilon_{p_\zeta}^\iota(v,\bar{p},\bar{n})\right],\Upsilon_n^\iota(v,\bar{p},\bar{n}),\varrho_{\Upsilon^\iota}\Big)\Big)dv$$

$$+\int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \bar{\lambda}_l\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(G_p^l(v,\bar{p},\bar{n})$$

$$-\frac{\partial}{\partial v^\zeta}\left[G_{p_\zeta}^l(v,\bar{p},\bar{n})\right],G_n^l(v,\bar{p},\bar{n}),\varrho_{U^l}\Big)\Big)dv$$

$$+\int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \bar{\pi}_j\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n}) - \frac{\partial}{\partial v^\zeta}\left[H_{p_\zeta}^j(v,\bar{p},\bar{p}_\zeta,\bar{n})\right],$$

$$H_n^j(v,\bar{p},\bar{p}_\zeta,\bar{n}),\varrho_{S^j}\Big)\Big)dv$$

$$+\int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} (-\bar{\pi}_j)\,\chi\Big(v,\tilde{p},\tilde{n},\bar{p},\bar{n};\Big(-H_p^j(v,\bar{p},\bar{p}_\zeta,\bar{n})$$

$$- \frac{\partial}{\partial v^{\zeta}} \left[-H^{j}_{p_{\zeta}}(v, \bar{p}, \bar{p}_{\zeta}, \bar{n}) \right],$$
$$- H^{j}_{n}(v, \bar{p}, \bar{p}_{\zeta}, \bar{n}), \varrho_{S^{j}} \Big) \Big) dv < 0.$$

The rest of the proof is similar as in Theorem 2. □

Let M, L be some subsets of E such that $M \cup L = E,\ M \cap L = \emptyset$, and let

$$\lambda_M(v)^T U^M(v, \gamma(v), \delta(v)) := \sum_{l \in M} \lambda_l(v) U^l(v, \gamma(v), \delta(v))$$

and

$$\lambda_L(v)^T U^L(v, \gamma(v), \delta(v)) := \sum_{l \in L} \lambda_l(v) U^l(v, \gamma(v), \delta(v)).$$

In this part of the study, we prove a duality connection between (NMEP) and its mixed multi-cost dual model (Dual) formulated as below:

$$\text{(Dual)} \quad \max_{(\gamma,\delta)} \int_{\Omega_{v_1,v_2}} \big(\Upsilon(v, \gamma(v), \delta(v)) + \lambda_M(v)^T U^M(v, \gamma(v), \delta(v)) e \big)\, dv$$

subject to

$$\eta^T \Upsilon_p(v, \gamma(v), \delta(v)) + \lambda(v)^T G_p(v, \gamma(v), \delta(v)) + \pi(v)^T H_p(v, \gamma(v), \dot{\gamma}(v), \delta(v))$$
$$= \frac{\partial}{\partial v^{\zeta}} \Big[\eta^T \Upsilon_{p_{\zeta}}(v, \gamma(v), \delta(v)) + \lambda(v)^T G_{p_{\zeta}}(v, \gamma(v), \delta(v))$$
$$+ \pi(v)^T H_{p_{\zeta}}(v, \gamma(v), \dot{\gamma}(v), \delta(v)) \Big],$$
$$\eta^T \Upsilon_n(v, \gamma(v), \delta(v)) + \lambda(v)^T G_n(v, \gamma(v), \delta(v)) + \pi(v)^T H_n(v, \gamma(v), \dot{\gamma}(v), \delta(v)) = 0,$$
$$\int_{\Omega_{v_1,v_2}} \lambda_L(v)^T U^L(v, \gamma(v), \delta(v)) dv \geq 0,$$
$$\int_{\Omega_{v_1,v_2}} \pi(v)^T S(v, \gamma(v), \dot{\gamma}(v), \delta(v)) dv = 0,$$
$$\gamma(v_1) = p_1,\ \gamma(v_2) = p_2,\ \eta \geq 0,\ \eta^T e = 1,\ \lambda(v) \geqq 0,\ v \in \Omega_{v_1,v_2},$$

where $e = (1, 1, \ldots, 1) \in \mathbb{R}^z$ is a z-dimensional vector.

Remark 1. For $L = \emptyset$, we obtain the Wolfe dual model. If $M = \emptyset$, we get the Mond-Weir dual model.

Further, we consider Ω_V is the feasible solution set $(\gamma, \delta, \eta, \lambda, \pi)$ of (Dual). We denote by Y the set

$$Y = \{ (\gamma, \delta) \in B \times C : (\gamma, \delta, \eta, \lambda, \pi) \in \Omega_V \}.$$

Theorem 4. *Let (p,n) and $(\gamma,\delta,\eta,\lambda,\pi)$ be any feasible solutions of (NMEP) and (Dual), respectively. In addition, we consider the statements are fulfilled:*

(a) $\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,p,n)dv, \iota\in H$, *is strictly* $(\chi,\varrho_{\Upsilon^\iota})$*-invex at* (γ,δ) *on* $V\cup Y$*;*

(n) $\int_{\Omega_{v_1,v_2}} U^l(v,p,n)dv, l\in E$, *is* (χ,ϱ_{U^l})*-invex at* (γ,δ) *on* $V\cup Y$*;*

(c) $\int_{\Omega_{v_1,v_2}} S^j(v,p,p_\zeta,n)dv, j\in A^+(v)$, *is* (χ,ϱ_{S^j})*-invex at* (γ,δ) *on* $V\cup Y$*;*

(d) $-\int_{\Omega_{v_1,v_2}} S^j(v,p,p_\zeta,n)dv, j\in A^-(v)$, *is* (χ,ϱ_{S^j})*-invex at* (γ,δ) *on* $V\cup Y$*;*

(e) $\sum_{\iota=1}^{z}\eta_\iota\varrho_{\Upsilon^\iota}+\sum_{l=1}^{x}\lambda_\iota\varrho_{U^l}+\sum_{j\in A^+(v)}\kappa_j\varrho_{S^j}-\sum_{j\in A^-(v)}\kappa_j\varrho_{S^j}\geq 0.$

Then, the relations cannot hold

$$\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,p,n)dv\leq\int_{\Omega_{v_1,v_2}}\left(\Upsilon^\iota(v,\gamma,\delta)+\lambda_M(v)^TU^M(v,\gamma,\delta)\right)dv,\ \iota\in H, \tag{37}$$

and

$$\int_{\Omega_{v_1,v_2}} \Upsilon^s(v,p,n)dv<\int_{\Omega_{v_1,v_2}}\left(\Upsilon^s(v,\gamma,\delta)+\lambda_M(v)^TU^M(v,\gamma,\delta)\right)dv, \tag{38}$$

for some $s\in H$*.*

Proof. By contradiction, we assume that the relations given in (37) and (38) are valid. By considering the assumptions in (a)-(d), we get

$$\begin{aligned}
&\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,p,n)dv-\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,\gamma,\delta)dv\\
&>\int_{\Omega_{v_1,v_2}}\chi\Big(v,p,n,\gamma,\delta;\Big(\Upsilon^\iota_p(v,\gamma,\delta)\\
&-\frac{\partial}{\partial v^\zeta}\left[\Upsilon^\iota_{p_\zeta}(v,\gamma,\delta)\right],\Upsilon^\iota_n(v,\gamma,\delta),\varrho_{\Upsilon^\iota}\Big)\Big)dv,\ \iota\in H, \qquad (39)\\
&\int_{\Omega_{v_1,v_2}} U^l(v,p,n)dv-\int_{\Omega_{v_1,v_2}} U^l(v,\gamma,\delta)dv
\end{aligned}$$

$$\geq \int_{\Omega_{v_1,v_2}} \chi\Big(v,p,n,\gamma,\delta;\Big(G^l_p(v,\gamma,\delta)$$
$$- \frac{\partial}{\partial v^\zeta}\Big[G^l_{p_\zeta}(v,\gamma,\delta)\Big], G^l_n(v,\gamma,\delta), \varrho_{U^l}\Big)\Big)dv,\ l\in E, \tag{40}$$
$$\int_{\Omega_{v_1,v_2}} S^j(v,p,p_\zeta,n)dv - \int_{\Omega_{v_1,v_2}} S^j(v,\gamma,\dot{\gamma},\delta)dv$$
$$\geq \int_{\Omega_{v_1,v_2}} \chi\Big(v,p,n,\gamma,\delta;\Big(H^j_p(v,\gamma,\dot{\gamma},\delta) - \frac{\partial}{\partial v^\zeta}\Big[H^j_{p_\zeta}(v,\gamma,\dot{\gamma},\delta)\Big],$$
$$H^j_n(v,\gamma,\dot{\gamma},\delta), \varrho_{S^j}\Big)\Big)dv,\ j\in A^+(v), \tag{41}$$
$$-\int_{\Omega_{v_1,v_2}} S^j(v,p,p_\zeta,n)dv + \int_{\Omega_{v_1,v_2}} S^j(v,\gamma,\dot{\gamma},\delta)dv$$
$$\geq \int_{\Omega_{v_1,v_2}} \chi\Big(v,p,n,\gamma,\delta;\Big(-H^j_p(v,\gamma,\dot{\gamma},\delta) - \frac{\partial}{\partial v^\zeta}\left[-S^j(v,\gamma,\dot{\gamma},\delta)\right],$$
$$-H^j_n(v,\gamma,\dot{\gamma},\delta), \varrho_{S^j}\Big)\Big)dv,\ j\in A^-(v), \tag{42}$$

are valid. Since $\lambda(v) \geqq 0$, then (40) gives

$$\int_{\Omega_{v_1,v_2}} \lambda_l U^l(v,p,n)dv - \int_{\Omega_{v_1,v_2}} \lambda^l U^l(v,\gamma,\delta)dv$$
$$\geq \int_{\Omega_{v_1,v_2}} \lambda_l\chi\Big(v,p,n,\gamma,\delta;\Big(G^l_p(v,\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\Big[G^l_{p_\zeta}(v,\gamma,\delta)\Big],$$
$$G^l_n(v,\gamma,\delta), \varrho_{U^l}\Big)\Big)dv,\ l\in E. \tag{43}$$

Using the feasibility of (p,n) and $(\gamma,\delta,\eta,\lambda,\pi)$ in problems (NMEP) and (Dual), respectively, we get

$$-\int_{\Omega_{v_1,v_2}} \sum_{l\in M} \lambda_l U^l(v,\gamma,\delta)dv$$
$$\geq \int_{\Omega_{v_1,v_2}} \sum_{l\in M} \lambda_l\chi\Big(v,p,n,\gamma,\delta;\Big(G^l_p(v,\gamma,\delta)$$
$$- \frac{\partial}{\partial v^\zeta}\Big[G^l_{p_\zeta}(v,\gamma,\delta)\Big], G^l_n(v,\gamma,\delta), \varrho_{U^l}\Big)\Big)dv \tag{44}$$

and

$$\int_{\Omega_{v_1,v_2}} \sum_{l\in L} \lambda_l\chi\Big(v,p,n,\gamma,\delta;\Big(G^l_p(v,\gamma,\delta)$$
$$-\frac{\partial}{\partial v^\zeta}\Big[G^l_{p_\zeta}(v,\gamma,\delta)\Big], G^l_n(v,\gamma,\delta), \varrho_{U^l}\Big)\Big)dv \leq 0. \tag{45}$$

We use the relations in (39), (44) and (45), involving

$$\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,p,n)dv - \int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,\gamma,\delta)dv - \int_{\Omega_{v_1,v_2}} \sum_{l\in M} \lambda_l U^l(v,\gamma,\delta)dv$$
$$> \int_{\Omega_{v_1,v_2}} \chi\Big(v,p,n,\gamma,\delta; \Big(\Upsilon^\iota_p(v,\gamma,\delta)$$
$$- \frac{\partial}{\partial v^\zeta}\left[\Upsilon^\iota_{p_\zeta}(v,\gamma,\delta)\right], \Upsilon^\iota_n(v,\gamma,\delta), \varrho_{\Upsilon^\iota}\Big)\Big)dv$$
$$+ \int_{\Omega_{v_1,v_2}} \sum_{l\in M} \lambda_l \chi\Big(v,p,n,\gamma,\delta; \Big(G^l_p(v,\gamma,\delta)$$
$$- \frac{\partial}{\partial v^\zeta}\left[G^l_{p_\zeta}(v,\gamma,\delta)\right], G^l_n(v,\gamma,\delta), \varrho_{U^l}\Big)\Big)dv$$
$$+ \int_{\Omega_{v_1,v_2}} \sum_{l\in L} \lambda_l \chi\Big(v,p,n,\gamma,\delta; \Big(G^l_p(v,\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\left[G^l_{p_\zeta}(v,\gamma,\delta)\right],$$
$$G^l_n(v,\gamma,\delta), \varrho_{U^l}\Big)\Big)dv, \ \iota \in H.$$

Taking into account the relation $M \cup L = E$, the above inequality produces

$$\int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,p,n)dv - \int_{\Omega_{v_1,v_2}} \Upsilon^\iota(v,\gamma,\delta)dv - \int_{\Omega_{v_1,v_2}} \sum_{l\in M} \lambda_l U^l(v,\gamma,\delta)dv$$

$$> \int_{\Omega_{v_1,v_2}} \chi\Big(v,p,n,\gamma,\delta; \Big(\Upsilon^\iota_p(v,\gamma,\delta)$$
$$- \frac{\partial}{\partial v^\zeta}\left[\Upsilon^\iota_{p_\zeta}(v,\gamma,\delta)\right], \Upsilon^\iota_n(v,\gamma,\delta), \varrho_{\Upsilon^\iota}\Big)\Big)dv$$
$$+ \int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \lambda_l \chi\Big(v,p,n,\gamma,\delta; \Big(G^l_p(v,\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\left[G^l_{p_\zeta}(v,\gamma,\delta)\right],$$
$$G^l_n(v,\gamma,\delta), \varrho_{U^l}\Big)\Big)dv, \ \iota \in H. \tag{46}$$

By relations in (37), (38) and (46), it results

$$\int_{\Omega_{v_1,v_2}} \chi\left(v,p,n,\gamma,\delta; \left(\Upsilon^\iota_p(v,\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\left[\Upsilon^\iota_{p_\zeta}(v,\gamma,\delta)\right], \Upsilon^\iota_n(v,\gamma,\delta), \varrho_{\Upsilon^\iota}\right)\right) dv$$
$$+ \int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \lambda_l \chi\Big(v,p,n,\gamma,\delta; \Big(G^l_p(v,\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\left[G^l_{p_\zeta}(v,\gamma,\delta)\right],$$
$$G^l_n(v,\gamma,\delta), \varrho_{U^l}\Big)\Big)dv < 0, \ \iota \in H. \tag{47}$$

Since $\eta \geq 0$ and $\eta^T e = 1$, then (47) gives

$$\int_{\Omega_{v_1,v_2}} \sum_{\iota=1}^{z} \eta_\iota \chi\Big(v,p,n,\gamma,\delta;\Big(\Upsilon^\iota_p(v,\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\left[\Upsilon^\iota_p(v,\gamma,\delta)\right], \Upsilon^\iota_n(v,\gamma,\delta), \varrho_{\Upsilon^\iota}\Big)\Big)dv + \int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \lambda_l \chi\Big(v,p,n,\gamma,\delta;\Big(G^l_p(v,\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\left[G^l_{p_\zeta}(v,\gamma,\delta)\right], G^l_n(v,\gamma,\delta), \varrho_{U^l}\Big)\Big)dv < 0. \tag{48}$$

Thus, relations given in (41) and (42) produce, respectively,

$$\int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \kappa_j S^j(v,p,p_\zeta,n)dv - \int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \kappa_j S^j(v,\gamma,\dot\gamma,\delta)dv \geq \int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \kappa_j \chi\Big(v,p,n,\gamma,\delta;\Big(H^j_p(v,\gamma,\dot\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\left[H^j_p(v,\gamma,\dot\gamma,\delta)\right], H^j_n(v,\gamma,\dot\gamma,\delta), \varrho_{S^j}\Big)\Big)dv \tag{49}$$

and

$$\int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} \kappa_j S^j(v,p,p_\zeta,n)dv - \int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} \kappa_j S^j(v,\gamma,\dot\gamma,\delta)dv \geq \int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} (-\kappa_j)\, \chi\Big(v,p,n,\gamma,\delta;\Big(-H^j_p(v,\gamma,\dot\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\left[-H^j_p(v,\gamma,\dot\gamma,\delta)\right], -H^j_n(v,\gamma,\dot\gamma,\delta), \varrho_{S^j}\Big)\Big)dv. \tag{50}$$

Adding both sides of (49) and (50), we get

$$\int_{\Omega_{v_1,v_2}} \sum_{j\in A} \kappa_j S^j(v,p,p_\zeta,n)dv - \int_{\Omega_{v_1,v_2}} \sum_{j\in A} \kappa_j S^j(v,\gamma,\dot\gamma,\delta)dv \geq \int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \kappa_j \chi\Big(v,p,n,\gamma,\delta;\Big(H^j_p(v,\gamma,\dot\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\left[H^j_p(v,\gamma,\dot\gamma,\delta)\right], H^j_n(v,\gamma,\dot\gamma,\delta), \varrho_{S^j}\Big)\Big)dv + \int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} (-\kappa_j)\, \chi\Big(v,p,n,\gamma,\delta;\Big(-H^j_p(v,\gamma,\dot\gamma,\delta) - \frac{\partial}{\partial v^\zeta}\left[-H^j_p(v,\gamma,\dot\gamma,\delta)\right], -H^j_n(v,\gamma,\dot\gamma,\delta), \varrho_{S^j}\Big)\Big)dv.$$

Hence, by the feasibility of the pair (p,n) and of $(\gamma,\delta,\eta,\lambda,\pi)$ in (NMEP) and (Dual), respectively, it results

$$\int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \kappa_j \chi\Big(v,p,n,\gamma,\delta;\Big(H_p^j(v,\gamma,\dot{\gamma},\delta)-\frac{\partial}{\partial v^\zeta}\left[H_p^j(v,\gamma,\dot{\gamma},\delta)\right],$$
$$H_n^j(v,\gamma,\dot{\gamma},\delta),\varrho_{S^j}\Big)\Big)dv$$
$$+\int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} (-\kappa_j)\,\chi\Big(v,p,n,\gamma,\delta;\Big(-H_p^j(v,\gamma,\dot{\gamma},\delta)-\frac{\partial}{\partial v^\zeta}\left[-H_p^j(v,\gamma,\dot{\gamma},\delta)\right],$$
$$-H_n^j(v,\gamma,\dot{\gamma},\delta),\varrho_{S^j}\Big)\Big)dv\leq 0. \tag{51}$$

Hence, the relations given in (48) and (51) yield

$$\int_{\Omega_{v_1,v_2}} \sum_{\iota=1}^{z} \eta_\iota \chi\Big(v,p,n,\gamma,\delta;\Big(\Upsilon_p^\iota(v,\gamma,\delta)$$
$$-\frac{\partial}{\partial v^\zeta}\Big[\Upsilon_{p_\zeta}^\iota(v,\gamma,\delta)\Big],\Upsilon_n^\iota(v,\gamma,\delta),\varrho_{\Upsilon^\iota}\Big)\Big)dv$$
$$+\int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \lambda_l \chi\Big(v,p,n,\gamma,\delta;\Big(G_p^l(v,\gamma,\delta)$$
$$-\frac{\partial}{\partial v^\zeta}\Big[G_{p_\zeta}^l(v,\gamma,\delta)\Big],G_n^l(v,\gamma,\delta),\varrho_{U^l}\Big)\Big)dv$$
$$+\int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \kappa_j \chi\Big(v,p,n,\gamma,\delta;\Big(H_p^j(v,\gamma,\dot{\gamma},\delta)-\frac{\partial}{\partial v^\zeta}\left[H_p^j(v,\gamma,\dot{\gamma},\delta)\right],$$
$$H_n^j(v,\gamma,\dot{\gamma},\delta),\varrho_{S^j}\Big)\Big)dv$$
$$+\int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} (-\kappa_j)\,\chi\Big(v,p,n,\gamma,\delta;\Big(-H_p^j(v,\gamma,\dot{\gamma},\delta)$$
$$-\frac{\partial}{\partial v^\zeta}\left[-H_p^j(v,\gamma,\dot{\gamma},\delta)\right],-H_n^j(v,\gamma,\dot{\gamma},\delta),\varrho_{S^j}\Big)\Big)dv<0. \tag{52}$$

We denote

$$\tilde{\eta}_\iota=\frac{\eta_\iota}{\sum_{\iota=1}^{z}\eta_\iota+\sum_{l=1}^{x}\lambda_l(v)+\sum_{j\in A^+(v)}\kappa_j(v)-\sum_{j\in A^-(v)}\kappa_j(v)},$$
$$\iota\in H, \tag{53}$$

$$\tilde{\lambda}_l(v) = \frac{\lambda_l(v)}{\sum_{\iota=1}^{z} \eta_\iota + \sum_{l=1}^{x} \lambda_l(v) + \sum_{j \in A^+(v)} \kappa_j(v) - \sum_{j \in A^-(v)} \kappa_j(v)},$$
$$l \in E, \tag{54}$$
$$\tilde{\pi}_j(v) = \frac{\kappa_j(v)}{\sum_{\iota=1}^{z} \eta_\iota + \sum_{l=1}^{x} \lambda_l(v) + \sum_{j \in A^+(v)} \kappa_j(v) - \sum_{j \in A^-(v)} \kappa_j(v)},$$
$$j \in A^+(v), \tag{55}$$

$$\tilde{\pi}_j(v) = \frac{-\kappa_j(v)}{\sum_{\iota=1}^{z} \eta_\iota + \sum_{l=1}^{x} \lambda_l(v) + \sum_{j \in A^+(v)} \kappa_j(v) - \sum_{j \in A^-(v)} \kappa_j(v)}, j \in A^-(v). \tag{56}$$

By (53) – (56), it follows that $0 \le \tilde{\eta}_\iota \le 1,\ \iota \in H$, but $\tilde{\eta}_\iota > 0$ for at least one $\iota \in H,\ 0 \le \tilde{\lambda}_l(v) \le 1, l \in E,\ 0 \le \tilde{\pi}_j(v) \le 1,\ j \in A$, and, moreover,

$$\sum_{\iota=1}^{z} \tilde{\eta}_\iota + \sum_{l=1}^{x} \tilde{\lambda}_l(v) + \sum_{j \in A^+(v)} \tilde{\pi}_j(v) + \sum_{j \in A^-(v)} \tilde{\pi}_j(v) = 1. \tag{57}$$

Combining (52)-(56), we get

$$\int_{\Omega_{v_1,v_2}} \sum_{\iota=1}^{z} \tilde{\eta}_\iota \chi\Big(v, p, n, \gamma, \delta; \Big(\Upsilon_p^\iota(v, \gamma, \delta) -$$
$$\frac{\partial}{\partial v^\zeta} \Big[\Upsilon_{p_\zeta}^\iota(v, \gamma, \delta)\Big], \Upsilon_n^\iota(v, \gamma, \delta), \varrho_{\Upsilon^\iota}\Big)\Big) dv$$
$$+ \int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \tilde{\lambda}_l \chi\Big(v, p, n, \gamma, \delta; \Big(G_p^l(v, \gamma, \delta)$$
$$- \frac{\partial}{\partial v^\zeta} \Big[G_p^l(v, \gamma, \delta)\Big], G_n^l(v, \gamma, \delta), \varrho_{U^l}\Big)\Big) dv$$
$$+ \int_{\Omega_{v_1,v_2}} \sum_{j \in A^+(v)} \tilde{\pi}_j \chi\Big(v, p, n, \gamma, \delta; \Big(H_p^j(v, \gamma, \dot{\gamma}, \delta) - \frac{\partial}{\partial v^\zeta} \left[H_p^j(v, \gamma, \dot{\gamma}, \delta)\right],$$
$$H_n^j(v, \gamma, \dot{\gamma}, \delta), \varrho_{S^j}\Big)\Big) dv$$
$$+ \int_{\Omega_{v_1,v_2}} \sum_{j \in A^-(v)} \tilde{\pi}_j \chi\Big(v, p, n, \gamma, \delta; \Big(-H_p^j(v, \gamma, \dot{\gamma}, \delta) - \frac{\partial}{\partial v^\zeta} \left[-H_p^j(v, \gamma, \dot{\gamma}, \delta)\right],$$
$$- H_n^j(v, \gamma, \dot{\gamma}, \delta), \varrho_{S^j}\Big)\Big) dv < 0. \tag{58}$$

By Definition 2, it follows that $\chi(v, p, n, \gamma, \delta, \cdot)$ is convex on $\mathbb{R}^{r+\omega+1}$. Thus,

since (57) is valid, then Definition 1 involves

$$\int_{\Omega_{v_1,v_2}} \sum_{\iota=1}^{z} \tilde{\eta}_\iota \chi\Big(v,p,n,\gamma,\delta;\Big(\Upsilon_p^\iota(v,\gamma,\delta)$$

$$-\frac{\partial}{\partial v^\zeta}\left[\Upsilon_{p_\zeta}^\iota(v,\gamma,\delta)\right],\Upsilon_n^\iota(v,\gamma,\delta),\varrho_{\Upsilon^\iota}\Big)\Big)\Big)dv$$

$$+\int_{\Omega_{v_1,v_2}} \sum_{l=1}^{x} \tilde{\lambda}_l \chi\Big(v,p,n,\gamma,\delta;\Big(G_p^l(v,\gamma,\delta)$$

$$-\frac{\partial}{\partial v^\zeta}\left[G_{p_\zeta}^l(v,\gamma,\delta)\right],G_n^l(v,\gamma,\delta),\varrho_{U^l}\Big)\Big)dv$$

$$+\int_{\Omega_{v_1,v_2}} \sum_{j\in A^+(v)} \tilde{\pi}_j \chi\Big(v,p,n,\gamma,\delta;\Big(H_p^j(v,\gamma,\dot{\gamma},\delta)-\frac{\partial}{\partial v^\zeta}\left[H_p^j(v,\gamma,\dot{\gamma},\delta)\right],$$

$$H_n^j(v,\gamma,\dot{\gamma},\delta),\varrho_{S^j}\Big)\Big)dv$$

$$+\int_{\Omega_{v_1,v_2}} \sum_{j\in A^-(v)} \tilde{\pi}_j \chi\Big(v,p,n,\gamma,\delta;\Big(-H_p^j(v,\gamma,\dot{\gamma},\delta)-\frac{\partial}{\partial v^\zeta}\left[-H_{p_\zeta}^j(v,\gamma,\dot{\gamma},\delta)\right],$$

$$-H_n^j(v,\gamma,\dot{\gamma},\delta),\varrho_{S^j}\Big)\Big)dv$$

$$\geq \int_{\Omega_{v_1,v_2}} \chi\Big(v,p,n,\gamma,\delta;\Big(\Big[\sum_{\iota=1}^{z}\tilde{\eta}_\iota\Upsilon_p^\iota(v,\gamma,\delta)+\sum_{l=1}^{x}\tilde{\lambda}_l G_p^l(v,\gamma,\delta)$$

$$+\sum_{j\in A^+(v)}\tilde{\pi}_j H_p^j(v,\gamma,\dot{\gamma},\delta)$$

$$+\sum_{j\in A^-(v)}\left(-\tilde{\pi}_j\right)H_p^j(v,\gamma,\dot{\gamma},\delta)\Big]-\frac{\partial}{\partial v^\zeta}\Big[\sum_{\iota=1}^{z}\tilde{\eta}_\iota\Upsilon_{p_\zeta}^\iota(v,\gamma,\delta)+\sum_{l=1}^{x}\tilde{\lambda}_l G_{p_\zeta}^l(v,\gamma,\delta)$$

$$+\sum_{j\in A^+(v)}\tilde{\pi}_j H_{p_\zeta}^j(v,\gamma,\dot{\gamma},\delta)+\sum_{j\in A^-(v)}\left(-\tilde{\pi}_j\right)H_{p_\zeta}^j(v,\gamma,\dot{\gamma},\delta)\Big],$$

$$\sum_{\iota=1}^{z}\tilde{\eta}_\iota\Upsilon_n^\iota(v,\gamma,\delta)+\sum_{l=1}^{x}\tilde{\lambda}_l G_n^l(v,\gamma,\delta)+\sum_{j\in A^+(v)}\tilde{\pi}_j H_n^j(v,\gamma,\dot{\gamma},\delta)$$

$$+\sum_{j\in A^-(v)}(-\tilde{\pi}_j)H_n^j(v,\gamma,\dot{\gamma},\delta),\sum_{\iota=1}^{z}\tilde{\eta}_\iota\varrho_{\Upsilon^\iota}$$

$$+\sum_{l=1}^{x}\tilde{\lambda}_l\varrho_{U^l}+\sum_{j\in A^+(v)\cup A^-(v)}\tilde{\pi}_j\varrho_{S^j}\Big)\Big)dv.$$

Combining (58) and the above relation, we have

$$\begin{aligned}\int_{\Omega_{v_1,v_2}} \chi\Big(v,p,n,\gamma,\delta;\Big(\Big[\sum_{\iota=1}^{z}\tilde{\eta}_\iota \Upsilon_p^\iota(v,\gamma,\delta)+\sum_{l=1}^{x}\tilde{\lambda}_l G_p^l(v,\gamma,\delta)\\ +\sum_{j\in A^+(v)}\tilde{\pi}_j H_p^j(v,\gamma,\dot{\gamma},\delta) \qquad (59)\\ +\sum_{j\in A^-(v)}(-\tilde{\pi}_j)\,H_p^j(v,\gamma,\dot{\gamma},\delta)\Big]-\frac{\partial}{\partial v^\zeta}\Big[\sum_{\iota=1}^{z}\tilde{\eta}_\iota \Upsilon_{p_\zeta}^\iota(v,\gamma,\delta)+\sum_{l=1}^{x}\tilde{\lambda}_l G_{p_\zeta}^l(v,\gamma,\delta)\\ +\sum_{j\in A^+(v)}\tilde{\pi}_j H_{p_\zeta}^j(v,\gamma,\dot{\gamma},\delta)+\sum_{j\in A^-(v)}(-\tilde{\pi}_j)\,H_{p_\zeta}^j(v,\gamma,\dot{\gamma},\delta)\Big],\\ \sum_{\iota=1}^{z}\tilde{\eta}_\iota \Upsilon_n^\iota(v,\gamma,\delta)+\sum_{l=1}^{x}\tilde{\lambda}_l G_n^l(v,\gamma,\delta)+\sum_{j\in A^+(v)}\tilde{\pi}_j H_n^j(v,\gamma,\dot{\gamma},\delta)\\ +\sum_{j\in A^-(v)}(-\tilde{\pi}_j)H_n^j(v,\gamma,\dot{\gamma},\delta),\sum_{\iota=1}^{z}\tilde{\eta}_\iota\varrho_{\Upsilon^\iota}+\sum_{l=1}^{x}\tilde{\lambda}_l\varrho_{U^l}\\ +\sum_{j\in A^+(v)\cup A^-(v)}\tilde{\pi}_j\varrho_{S^j}\Big)\Big)dv<0.\end{aligned}$$

Hence, the constraints of (Dual) yield

$$\int_{\Omega_{v_1,v_2}} \chi\Big(v,p,n,\gamma,\delta;\Big(0,0,\sum_{\iota=1}^{z}\tilde{\eta}_\iota\varrho_{\Upsilon^\iota}+\sum_{l=1}^{x}\tilde{\lambda}_l\varrho_{U^l}+\sum_{j\in A^+(v)\cup A^-(v)}\tilde{\pi}_j\varrho_{S^j}\Big)\Big)dv<0. \qquad (60)$$

From the assumption given in (e), we get that

$$\sum_{\iota=1}^{z}\tilde{\eta}_\iota\varrho_{\Upsilon^\iota}+\sum_{l=1}^{x}\tilde{\lambda}_l\varrho_{U^l}+\sum_{j\in A^+(v)\cup A^-(v)}\tilde{\pi}_j\varrho_{S^j}\geq 0. \qquad (61)$$

By Definition 2, we have that $\chi(v,p,n,\gamma,\delta;(0,0,\alpha))\geq 0$, with $\alpha\in\mathbb{R}_+$. Now, relation given in (61) implies

$$\int_{\Omega_{v_1,v_2}} \chi\Big(v,p,n,\gamma,\delta;\Big(0,0,\sum_{\iota=1}^{z}\tilde{\eta}_\iota\varrho_{\Upsilon^\iota}+\sum_{l=1}^{x}\tilde{\lambda}_l\varrho_{U^l}+\sum_{j\in A^+(v)\cup A^-(v)}\tilde{\pi}_j\varrho_{S^j}\Big)\Big)dv\geq 0$$

is valid, contradicting (60). □

4 Conclusions and further developments

The investigation of optimization problems has been one of the most researched and attractive topics. In this paper, we have studied and characterized the solution set of a non-convex multi-cost extremization problem. Concretely, we have established some existence results of solutions associated with this optimization model governed by invex, pseudoinvex, or quasiinvex multiple-integral-type functionals. Also, we have established a dual connection between the efficient point of the considered non-convex multi-cost extremization problem and the efficient solution of the corresponding dual model.

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